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EFFECT OF MODULATION ON THE ONSET OF THERMAL CONVECTION IN A ROTATING FLUID

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NOMENCLATURE

L, width of channel;
Pr, Prandtl number (ν/κ);
Ra, Rayleigh number ($g\alpha\,\Delta T\,L^3/\kappa\nu$);
t, time;
T, temperature;
$\bar{T}$, temperature in the basic state;
Ta, Taylor number ($4\Omega^2\,L^4/\nu^2$);
x, y, z, rectangular co-ordinates;
w, velocity perturbation in the z direction.

Greek symbols

α, coefficient of thermal expansion;
ε, amplitude of modulation;
ζ, z component of vorticity;
θ, perturbation temperature;
κ, thermal diffusivity;
Λ, measure of enhancement of stability defined in equation (13);
ν, kinematic viscosity;
ρ, density;
Ω, angular velocity;
ω, nondimensional frequency;
ω^*, dimensional frequency.

Subscripts

R, denotes a reference quantity;
c, critical value;
0, denotes the initial state.

INTRODUCTION

VENEZIAN [1] first demonstrated that a fluid, when heated from below in an oscillatory manner, can have a higher value of the critical Rayleigh number for the onset of convection than when the fluid is heated with fixed temperatures at the boundaries. Although Venezian considered only small values of the modulation amplitude and considered only the case of "free–free" boundary conditions, his results are in qualitative agreement with those of Rosenblat and Tanaka [2], who solved the same problem for finite values of the modulation amplitude and for the more realistic case of rigid wall boundary conditions. The results for low values of the frequency of modulation are questionable, however, because, as pointed out by Rosenblat and Herbert [3], a linear analysis might not be applicable.

This communication is a preliminary study of the case when the fluid is rotating. For small Prandtl numbers, convection can begin in an oscillatory manner (i.e. the principle of exchange of stabilities does not hold), and the modulation might be expected to have more of a resonant effect. However, we find that significant deviation occurs even for a Prandtl number of unity, which is the only value considered in this note. For a Prandtl number of unity, a simplified equation for the disturbance velocity results, but a meaningful result is still obtained.

THE BASIC EQUATIONS

A layer of fluid is contained between two infinite walls, a distance L apart, and rotates about the vertical direction with angular velocity Ω. At the lower wall, the temperature is

$$T|_{z=0} = T_R+\tfrac{1}{2}\Delta T+\varepsilon\Delta T\cos\omega^* t, \tag{1}$$

whereas at the upper wall, the temperature is

$$T|_{z=L} = T_R-\tfrac{1}{2}\Delta T. \tag{2}$$

A Boussinesq fluid is assumed with density

$$\rho = \rho_R[1-\alpha(T-T_R)]. \tag{3}$$

The conduction solution satisfies

$$\frac{\partial \bar{T}}{\partial t} = \kappa\frac{\partial^2 \bar{T}}{\partial z^2} \tag{4}$$

and can be expressed as

$$\bar{T} = T_R+\Delta T(L-2z)/2L+\varepsilon T_1(z,t), \tag{5}$$

where T_1 satisfies (4) and the periodic temperature conditions at the wall.

We now perturb the basic state by letting

$$T = \bar{T}(z,t)+\theta(x,y,z,t), \tag{6}$$

where θ is assumed to be small enough that a linear stability analysis is valid.

If we nondimensionalize in the usual manner, cf. Venezian [1], then, following the development in Chapter III, Section 25, of Chandrasekhar [4]*, we can obtain the perturbation equations in the form

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 w = Ra\nabla_1^2\theta-\frac{2\Omega L^2}{\nu}\frac{\partial\zeta}{\partial z} \tag{7}$$

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t}-\nabla^2\right)\zeta = \frac{2\Omega L^2}{\nu}\frac{\partial w}{\partial z} \tag{8}$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\theta = -\frac{\partial \bar{T}}{\partial z}w. \tag{9}$$

It is clear that the case $Pr = 1$ is rather special because we can readily combine equations (7–9) to give the following equation

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)^2\nabla^2 w+Ta\frac{\partial^2 w}{\partial z^2} = -Ra\frac{\partial \bar{T}}{\partial z}\nabla_1^2 w, \tag{10}$$

which is of second order in time, whereas the more general case ($Pr \neq 1$) is of third order in time. Although this might seem too special, it does allow comparison to be made directly to the non-rotating case, which is of the same order in time.

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*The centrifugal force is ignored here, which implies $(\Omega^2 R/g) \ll 1$, where R is a characteristic horizontal dimension. The effects of this force for the unmodulated case are discussed, for instance, by Homsy and Hudson [5].

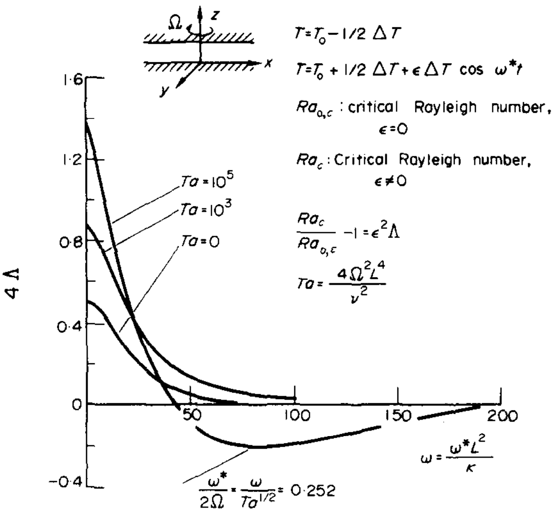

FIG. 1. Effect of modulation on the critical Rayleigh number.

For "free–free" surfaces, the boundary conditions are (cf. Chandrasekhar [4]),

$$w = \frac{\partial^2 w}{\partial z^2} = \frac{\partial^4 w}{\partial z^4} = 0 \quad \text{at} \quad z = 0, 1. \tag{11}$$

METHOD OF SOLUTION AND RESULTS

The basic parameters in the problem are the Rayleigh number Ra, the Taylor number Ta, the amplitude of modulation ε and the nondimensional frequency $\omega(= \omega^* L^2/\kappa)$, which results from the fact that we have used the usual method of nondimensionalizing the time on the basis of L and κ.

We assume that $\varepsilon \ll 1$, in the manner of Venezian [1] and expand in terms of ε, i.e. we let

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \tag{12a}$$

$$Ra = Ra_0 + \varepsilon Ra_1 + \varepsilon^2 Ra_2 + \ldots \tag{12b}$$

for fixed values of Ta and ω. The condition which determines the $Ra_j (j = 1, 2, \ldots)$ is simply that the solution exhibits no secular growth in time, i.e. that it is periodic in time. The theory is closely associated with that of the Mathieu equation, and the details are described fully in Venezian's paper [1]. The change in the Rayleigh number is then calculated for various ω for fixed values of Ta, and the procedure is repeated for various Ta.

It turns out that $Ra_1 = 0$, and therefore the natural presentation is in terms of

$$\varepsilon^2\Lambda = \frac{Ra_c}{(Ra_0)_c} - 1 = \varepsilon^2 \left(\frac{Ra_{2,c}}{Ra_{0,c}}\right), \tag{13}$$

where Λ is a measure of the enhancement of stability. The results are shown in Fig. 1. For $Ta = 0$, the curve agrees with that shown in Fig. 3 of Venezian [1], when we redefine his ε so as to agree with ours. The value for Λ decays monotonically as a function of ω, but it is positive, indicating that the critical Rayleigh number is increased by modulation. The same remains true for $Ta = 10^3$, although the value of Λ is greater and does not decay as rapidly with ω. For $Ta = 10^5$, however, Λ does not decay monotonically in a positive manner but becomes negative for $\omega > 42$, approximately. Hence, a lower value of the critical Rayleigh number results, with a maximum destabilization occurring for $\omega \approx 80$.

The results for $\omega \to 0$ are subject to the same criticism regarding use of linear analysis as in the case without rotation. For these larger values of ω at which destabilization occurs, however, this criticism cannot be made.

It is therefore of interest to ask how the mechanism of instability comes about. A rotating fluid can exhibit inertial waves, with a characteristic dimensional frequency no greater than 2Ω (cf. Chapter III, Section 23, of Chandrasekhar [4]). The nondimensional frequency can be expressed as

$$\omega = \frac{\omega^* L^2}{\kappa} = \left(\frac{\omega^*}{2\Omega}\right) Ta^{\frac{1}{2}} Pr \tag{14}$$

so, for $Pr = 1$, we have the result

$$\frac{\omega^*}{2\Omega} = \frac{\omega}{Ta^{\frac{1}{2}}}. \tag{15}$$

As shown in Fig. 1, the maximum destabilization occurs for $(\omega^*/2\Omega) = 0{\cdot}25$, i.e. the modulation frequency is within the range of inertial wave frequencies. It is therefore suggested that the destabilization can be associated with the incipient excitation of inertial waves. The maximum amount of destabilization for $Ta = 10^5$ is roughly of the same order as the maximum amount of stabilization achieved in the quasi-steady limit, for $Ta = 0$, suggested by Rosenblat and Herbert [3], e.g. see their Fig. 4 for $\varepsilon = 0{\cdot}1$.

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